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Mobile Sensor Placement for the 3-Coverage Problem in Confined Geometries

Haoyang Lyu, William Chang, Adrien Weihs
Department of Mathematics
University of California, Los Angeles
Los Angeles, CA, 90095, USA

Andrea L. Bertozzi*
Member, IEEE
Departments of Mathematics and
Mechanical and Aerospace Engineering
University of California, Los Angeles
Los Angeles, CA, 90095, USA

P. Jeffrey Brantingham
Department of Anthropology
University of California, Los Angeles
Los Angeles, CA, 90095, USA

***Abstract*—We consider the 3-coverage problem in confined geometries using mobile sensors. Many sensor problems require the use of multiple cooperative sensors for geolocation of events (e.g. using sound for triangulation). Optimal placement of sensors can be challenging, especially when confined to unusual geometries. We propose a novel swarming algorithm involving pairwise interaction potentials. Such models have low-energy cooperative states in which the agents form local Voronoi-style hexagonal patterns. We focus on the Morse potential and show that under certain choices of parameters, particles reach a desirable uniform distribution of 3-coverage. Such properties include (a) maintaining optimal distance as sensors are added or removed, (b) ability to navigate with limited communication between sensors, such as near-neighbor interactions for the graph topology of the sensor network, and (c) ability to target specific areas within a confined geometry especially with limited sensor resources.**

***Index Terms*—3-coverage, mobile sensors, pairwise interaction**

I. Introduction

In an increasingly interconnected and data-driven world, sensors serve as the foundational interface between physical phenomena and digital intelligence. These devices, which are capable of detecting and responding to environmental stimuli, have become indispensable in a wide spectrum of applications, including monitoring of climate change, optimizing industrial processes, and visual surveillance. As an example, the Denver Office of Community Violence Solutions (OCVS) gunshot detection program expects to install gunshot sensors around local neighborhoods that offer guidance to optimize the limited resources of community crime intervention [17]. Inspired by the program, this paper investigates the optimal sensor placement problem in a bounded domain with realistic constraints. As it is a well-established problem, many studies have proposed various strategies to address the optimal sensor placement problem [1], [2], [11], especially the 1-coverage problem, where it is expected that every point within the target area should be covered by 1 sensor. However, because the sensors adopt a Time Difference of Arrival mechanism for sound detection, 1-coverage is not sufficient for this application. Instead, a 3-coverage is required for the sensors to function as expected. Oriented by [14], we seek an optimal sensor packing pattern as a triangulation of the target area. Rather than approaching the triangulation problem geometrically as in [14], the present paper develops a novel approach by modeling the sensor placement procedure as a particle diffusion problem.

* denotes corresponding author (bertozzi@ucla.edu). This work is funded by US National Science Foundation grants 2125319 and 2152717.

II. Traditional Approach: Hexagonal Coverage

The triangulation coverage, as shown in Figure. 1, is known to be optimal for 1-coverage [13].

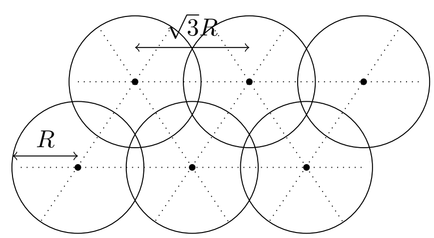

Fig. 1: Optimal Placement for 1-Coverage

And for 3-coverage, a natural extension is to overlay three such lattices, with each layer offset so that, ideally, every point lies in the intersection of three disks. However, when a *minimum separation requirement* $D_{\min}^{\mathrm{req}}$ is imposed—-so that no two sensors are placed closer than $D_{\min}^{\mathrm{req}}$—-this strategy can fail.

Instead, [14] proposed an optimal placement strategy by the use of TREs (Triple-Rounded-Edge). This method systematically patches triple-covered regions (TREs) where all

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covering sensors are at least $D_{\min}^{\mathrm{req}}$ apart, ensuring true 3-coverage throughout the region. According to [14], under a minimum possible sensor separation parameter $d \geq 0.196R$, full 3-coverage is guaranteed under separation constraints.

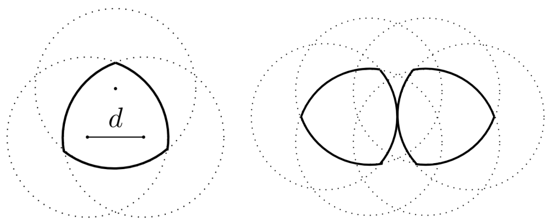


Fig. 2: Left: A TRE Shape; Right: Patching of 2 TREs for Optimal 3-Coverage

However, when optimizing the coverage of a *constrained* space, naive hexagonal placement can fail to guarantee full 3-coverage near physical boundaries. As illustrated in Figure. 3, placing sensors at the centers of hexagons yields a regular coverage pattern, but **points inside each hexagon (such as the red point in the figure) may not be contained within any of the TREs**. In the unconstrained case, every point is within a hexagon where *all* the vertices have sensors. However, in the constrained case, points near the boundary may only have proximity to one or two sensors, depending on sensor placement and radius. The blue line in the figure illustrates the location of a boundary to demonstrate this principle.

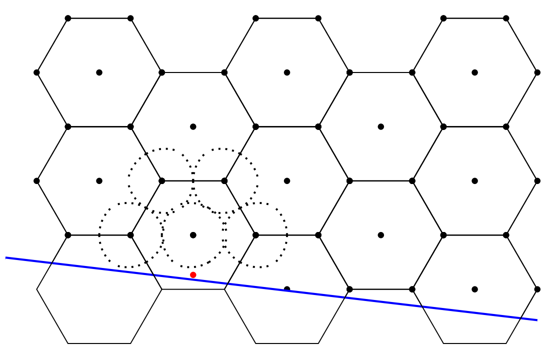


Fig. 3: The blue line is the region boundary, with the interior above the line. Black dots mark sensor TRE centers in a hexagonal tiling, and dotted circles show their coverage. While this arrangement gives triple coverage in the interior, the red dot—located inside a hexagon—shows that near the boundary, some points are not covered by three sensors. Specifically, the red point is not within any TRE shape, illustrating how boundary constraints can lead to insufficient multi-coverage even when using an optimal tiling.

III. Alternative approach - swarming model

We consider a pairwise interaction model for sensor node placement because, for certain potentials, the ground state exhibits quasi-hexagonal packing of particles. Boundary constraints can be added in a natural way as barriers to the motion of sensors. We explore the dynamics of such models using a gradient descent algorithm for the positioning x_i of the sensors:

$$\dot{x}_i = -\sum_{j \neq i} \nabla W(x_i - x_j) \tag{1}$$

This is an example of a first order discrete model for particle interactions. Mathematical models for swarming, schooling, and other aggregative behavior in biological systems have served as powerful tools to understand collective motion and emergent pattern formation in nature [8], [12], [18], [25], [26]. Many of these models capture how individuals interact through short-range repulsion and long-range attraction, leading to rich and sometimes unexpected spatial structures [4], [6], [9], [10], [15], [29], [30]. A central feature of these systems is that the positioning dynamics involves non-local interactions [3], [8], [16].

The **Morse potential**, introduced by Philip M. Morse in 1929 [20], is a mathematical model to describe the potential energy of a diatomic molecule. It improves upon the harmonic oscillator by accounting for the anharmonicity of real molecular vibrations and the dissociation of bonds. This potential has played a foundational role in quantum chemistry and molecular physics, particularly for its analytic solvability in the Schrödinger equation and its realistic depiction of vibrational spectra. The Morse potential takes the form of

$$W(r) = -C_a e^{-r/\ell_A} + C_r e^{-r/\ell_R}, \tag{2}$$

where the A terms correspond to attraction and the R terms correspond to repulsion. We choose this potential because there are broad parameter ranges in which the ground-state solution has fairly uniform coverage in a localized region. The work [7] considers a second order flocking model using the Morse potential with additional self-propulsion and drag the the particles. Here we propose to use the same potential, but in a first order model, so that the particles converge in time to a pattern that will cover a region of interest with boundaries.

IV. Hamiltonian Stability of the Morse Potential

Understanding the stability and emergent behavior of particle systems under long-range attraction has been a central theme in the study of self-organizing dynamics. In particular, [7] introduced a class of swarming models where particles interact via a combination of short-range repulsion and long-range attraction, modeled through a generalized Morse potential. A key outcome of their work is the delineation between *H-stable* systems, which remain dispersed as the number of particles grows, and *catastrophic* systems, where particles collapse into dense clusters. This classification hinges on whether the total interaction energy scales linearly or quadratically with particle number, and is closely tied to the sign of the spatial integral of the pairwise potential.

Newtonian interaction, which is purely attractive, falls squarely in the catastrophic regime: it lacks a characteristic length scale and fails the conditions for H-stability. Consequently, when particles are confined to a compact domain and interact via the Newtonian potential, they exhibit a collective drift toward concentration. However, in settings where the density is held fixed and the support of the mass is constrained to lie within a convex container, the nonlocal attraction generates an outward-pointing velocity at the boundary. This expansion pressure reflects the system's intrinsic instability and resembles the collapse dynamics observed in catastrophic regimes of the Morse potential. Our goal is to isolate and rigorously verify this boundary behavior in a deterministic setting, under the assumption of constant density and smooth evolution.

Theorem 1 (Hamiltonian Stability of the Generalized Morse Potential in all of Space)**.** *Let N identical particles in $\mathbb{R}^d$ (d-dimensional space) interact via the pairwise potential*

$$\phi(r) = C_r\, e^{-r/\ell_r} - C_a\, e^{-r/\ell_a},$$

where $C_r, \ell_r > 0$ denote the strength and range of the repulsion and $C_a, \ell_a > 0$ denote the strength and range of the attraction. Define the total potential energy

$$U \;=\; \sum_{1\le i<j\le N} \phi\big(|x_i - x_j|\big).$$

*Then the system is H-*stable *(i.e. there exists $B \ge 0$ such that $U \ge -B\,N$ for all configurations and all N) if and only if*

$$C_r\,\ell_r^d \;\ge\; C_a\,\ell_a^d.$$

Equivalently, letting

$$C \;=\; \frac{C_r}{C_a}, \qquad \ell \;=\; \frac{\ell_r}{\ell_a},$$

one has the stability criterion

$$C\,\ell^d \;\ge\; 1.$$

Proof. The total energy is

$$U = \sum_{1\le i<j\le N} \phi(|\vec{x}_i - \vec{x}_j|).$$

A system is *H-stable* if $U \ge -BN$ for some $B \ge 0$ and all N [24], ensuring energy per particle is bounded below as $N \to \infty$.

In the large-N (mean-field) limit,

$$U \approx \binom{N}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \phi(|\vec{x} - \vec{y}|)\rho(x)\rho(y)\,dx\,dy,$$

and for uniform density over a volume V,

$$U \approx \frac{N^2}{V^2}\int_{\mathbb{R}^d} \phi(r)\,dr.$$

Thus, H-stability requires

$$\int_{\mathbb{R}^d} \phi(r)\,dr \ge 0.$$

Switching to spherical coordinates,

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(r)\,d^d r &= S_d \int_0^\infty \left(C_r e^{-r/\ell_r} - C_a e^{-r/\ell_a}\right) r^{d-1} dr \\ &= S_d \Gamma(d)\left(C_r \ell_r^d - C_a \ell_a^d\right),\end{aligned}$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ and $\Gamma(d)$ is the gamma function.

Therefore, H-stability holds if and only if

$$C_r \ell_r^d \ge C_a \ell_a^d \qquad (\star)$$

or, equivalently, setting $C = \frac{C_r}{C_a}$ and $\ell = \frac{\ell_r}{\ell_a}$,

$$C\ell^d \ge 1.$$

□

V. Gradient Flow Stability

Both the model introduced by D'Orsogna et al. [7] and the framework of Bertozzi et al. [5] describe the collective dynamics of self-propelled particles interacting through pairwise potentials, but approach the problem from slightly different formalisms. In [7], each particle obeys a second-order ODE with self-propulsion, friction, and a pairwise Morse potential:

$$m\frac{d\vec{v}_i}{dt} = (\alpha - \beta|\vec{v}_i|^2)\vec{v}_i - \nabla_i U(\vec{x}_i) \tag{3}$$

where the interaction energy U is the sum of Morse potentials over all pairs. This second order model has both flocking solutions and rotating mills. The flocking solutions arise with each particle traveling with the same vector velocity $\vec{v}_0$ in which $|\vec{v}_0| = \sqrt{\alpha/\beta}$. These are traveling wave solutions and the relative configuration of the particles also specifies a steady-state solution of a first order system, which can be derived as a gradient flow of the potential.

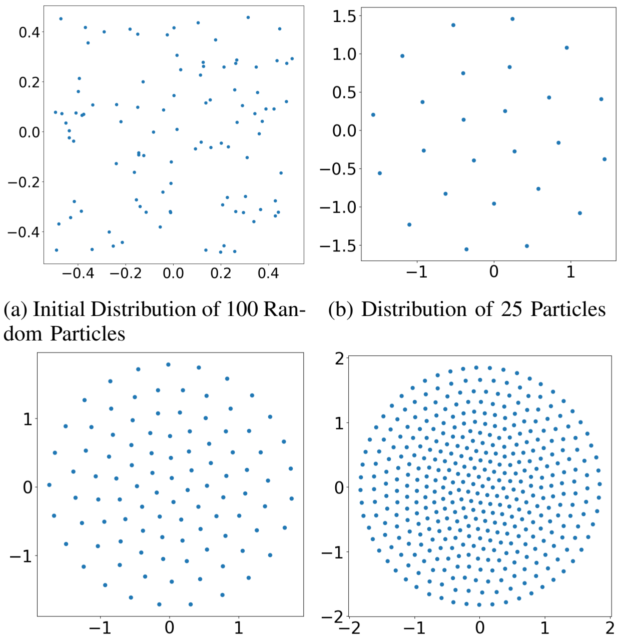
Such gradient flow dynamics are studied in [5], using the formula

$$\frac{dx_k}{dt} = \frac{1}{N}\sum_{j\neq k} \frac{-W'(|x_k - x_j|)}{|x_k - x_j|}(x_k - x_j). \tag{4}$$

with the potential W is defined to be the Morse potential (2). Note there is a different scaling here by a factor of $2N$ however that can be absorbed into the time scale with the same dynamics and equilibria as a gradient flow in the notation of [7]. We adapt (4) to the problem with boundaries for optimal sensor placement, below.

VI. Gradient Descent in Free Space and in Simple Bounded Domains

In this section, we present some numerical simulations of particle swarming with pairwise Morse potentials, both in free space and in idealistic bounded domains. In both cases we find equilibrium configurations with near-hexagonal orientation suggesting that they could work well for the 3-coverage problem. These simulations motivate the application of the model onto a more realistic and complex-shaped domain.



(a) Initial Distribution of 100 Random Particles (b) Distribution of 25 Particles

(c) Distribution of 100 Particles (d) Distribution of 400 Particles

Fig. 4: Equilibrium distribution of particles under a Morse potential (2) with parameters $C_a = 0.5$, $C_r = 1$, $l_a = 2$, $l_r = 0.5$ at time $T = 5000$.

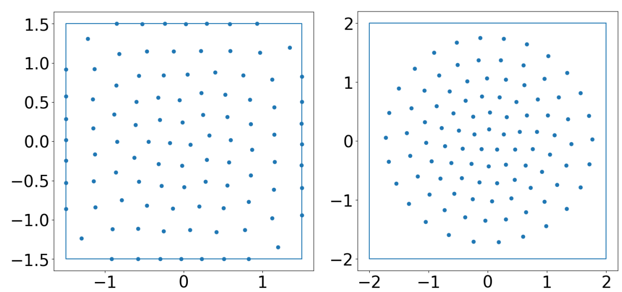
A. *Morse Potential with No Boundary*

We first simulate the diffusion of particles on a free space with parameters chosen such that uniform distribution of particles is reached at equilibrium. Particles are randomly placed in a 1×1 square centered around $(0, 0)$ and evolve according to Equation 4 without any boundary effects. The results are shown in Fig. 4. It is observed that, under certain parameter choices, the particles form a uniformly distributed disk with a quasi-hexagonal close packing structure, with a radius independent of total number N. This feature makes the model of interest to the problem in confined geometries.

B. *Simulation on a Bounded Domain*

To illustrate that particles remain evenly distributed when a boundary is imposed, we simulate the swarming of 100 particles on a bounded domain. Using the same initial condition and choice of parameters as in Fig. 4, the particles are now confined to a square with side length 1.5 or 2. The side lengths are specifically chosen so that when moving on a free space, the equilibrium distribution of particles will be slightly larger than a 1.5 by 1.5 domain, while completely in a 2 by 2 square. When pushed out of the domain, particles are projected back to the nearest point on the boundary in the square. This feature enables particles to move along the boundary and even flow back into the interior should the evolution equation dictate such behavior.

As shown in Fig. 5, the confined system evolves to a nearly uniform distribution. And when restricted to a larger domain, the final state ignores the boundary. This observation demonstrates that the final distribution of particles is not largely affected by the presence of a boundary. In the smaller square the particles reach the boundary and spread out evenly to cover the edge of the domain. Visual inspection suggests that coverage is robust except possibly for the far corners of the square. If the region is too large for full coverage as in the right image, the particles will self-organize into a roughly circular region with good coverage of the part of the domain they occupy. We note that such structured self-organization is desirable for the 3-coverage problem. Should some particles spread out further, their sensing range would not overlap with other particles, thus making them obsolete for 3-coverage needed for triangulation. This property inspires the use of the Morse potential for a more complex domain.



(a) Distribution of 100 Particles on a Square with Side Length 1.5. (b) Distribution of 100 Particles on a Square with Side Length 2.

Fig. 5: Equilibrium distribution of particles in a bounded domain at $T = 5000$ with Morse potential (2) parameters $C_a = 0.5$, $C_r = 1$, $l_a = 2$, $l_r = 0.5$.

VII. An Application to Gunshot Sensors

As a proof of concept, we consider a bounded domain in the shape of the city of Denver, motivated by the coverage problem discussed in [17]. We apply a two-step process of first smoothing out some small scale irregularity on the boundary for better flow of sensors along the boundary and then use the Morse potential swarming model for an optimal coverage.

A. *Preprocessing the Boundary*

The boundary of Denver is a highly irregular and non-convex shape. Therefore, we applied a spline interpolation with a smoothing factor of 0.1, so that major features of Denver map are preserved, while particles may better dissipate on the less irregular boundary. Moreover, since the geographical data are in longitude and latitude, to avoid numerical precision problems, we rescaled the side length in each axis by 10. The rescale only affects the characteristic length-scale of the sensor placement, without hindering the overall utility of the model.

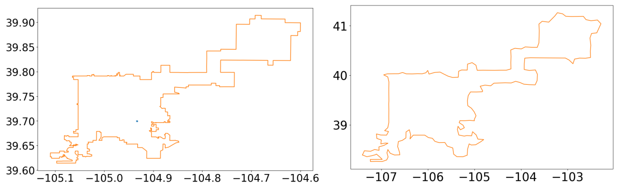
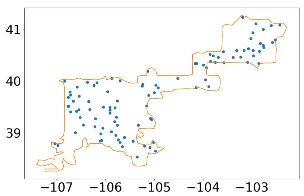
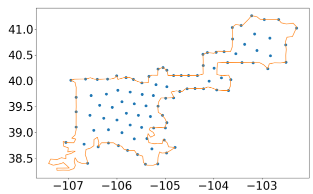


Fig. 6: Left: The Map of Denver; Right: The Preprocessed Boundary after Scaling by 10 and Smoothing by Spline Interpolation.

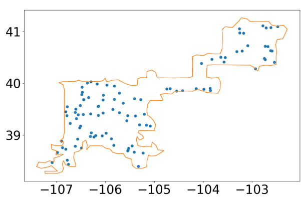
B. Simulation on the Map of Denver



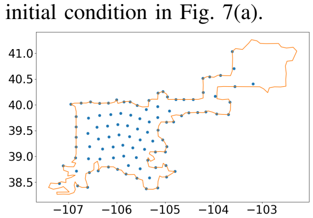
(a) Example of an initial random distribution for Denver.



(b) Sensor distribution at time $T = 5000$, after starting with initial condition in Fig. 7(a).



(c) Another instance of an initial random distribution that led to insufficient coverage



(d) Sensor distribution at time $T = 5000$ with initial condition from Figure. 7(c).

Fig. 7: Distribution of sensors inside the city of Denver under a gradient flow (1) with the Morse potential (2) with parameters $C_a = 0.1$, $C_r = 1$, $l_a = 1$, $l_r = 0.15$.

In Figure. 7(a), 100 sensors are randomly initiated within the city of Denver. Figure. 7(b) and (d) show final states using the same parameters but with different initial placements of sensors. Despite the non-convexity of domain, sensors can learn the bottleneck shape. However, whether the sensors achieve a uniform final state still depends on initial conditions, a situation we trace to the interaction between random initial sensor placement and the irregular boundary shape. Thus, it is possible to observe incomplete coverage as in Figure 7(d). It would be interesting to better understand the connection between the initial sensor placement and the final configuration through a more systematic study. Such differences are expected in a highly non-convex optimization problem such as this one. Nevertheless, we observe that with an appropriate distribution of initial placements in different parts of the domain, we can obtain the results seen in Fig. 7(b), exhibiting very uniform coverage.

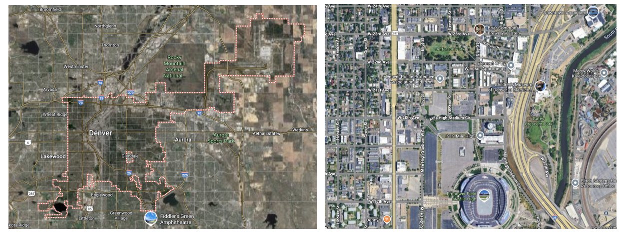


Fig. 8: (left) Satellite image of Denver (Google Maps) and (right) blowup of Downtown area showing details of streets, housing and businesses. Detailed images such as these can be used for data mining of fine scale accessible sensor placement locations.

VIII. Discussion and Future Work

In this paper, we demonstrate a proof of concept for the uniform distribution of particles under a gradient flow model with pairwise Morse potential in an irregular domain. This model could serve as a novel approach to the 3-coverage sensor placement problem. Potential future refinements to the present model are suggested below.

A. Stability Analysis with Parameter Choice

We have demonstrated through simulation that the Morse potential has parameters that allow for good coverage of a complex shape with boundaries, while still maintaining a relatively uniform distributed steady state. However, in second order systems such as studied in [7], one can observe both catastrophic collapse as well as H-stability, with a multitude of ground states. Therefore, we would like to further investigate the model and its role in the 3-coverage problem.

B. Role of Network Topology

There is a body of literature that considers complex graph interaction topologies of swarming agents with interesting features. Some simpler examples arise in consensus models in the control theory literature [19], [21]–[23], [32] with a very simple linear interaction control corresponding to W above in the form of a spring potential.

For potentials with both attraction and repulsion terms and nonlinear gradients, such as the Morse potential, there are fewer studies addressing the role of network topology for the pairwise interactions. One example already studied is the case of a compromise model in which the lowest energy state has all particles in one of a finite number of positions depending on the spatial dimension. For that model, one can find a critical probability of interaction below which the compromise model destabilizes [27], [28]. That work considers linear stability and uses random matrix theory to prove a rigorous result.

C. Role of Multiscale Geometry

Urban environments have multiscale geometry. The current model uses only the city boundary with some smoothing near highly non-convex regions. However, urban geography can also be understood on the scale of buildings and streets as well as densities of other measurable features such as population or crime. These features likely impact sensor placement decisions. For example, a sensor cannot be placed in the middle of a street. In the case of community-led programs, sensors must be affixed to buildings that have some investment or interest in the program, which could be local businesses or residences that agree to host sensors. More work needs to be done to incorporate multiscale geometry into the problem, including 3D effects from tall buildings and sound reverberation. Auxiliary data mining may also play an important role in refining the model (see e.g. Fig. 8). For example, we could consider the non-local means graph used in [31] for crime density estimation. The sensor placement shown in Fig. 7 is only a rough approximation of the true optimal distribution when multiscale geometric constraints are imposed.

REFERENCES

[1] V. Akbarzadeh, J.-C. Lévesque, C. Gagné, and M. Parizeau. Efficient sensor placement optimization using gradient descent and probabilistic coverage. *Sensors*, 14(8):15525–15552, 2014.

[2] X. Bai, S. Kumar, D. Xuan, Z. Yun, and T. H. Lai. Deploying wireless sensors to achieve both coverage and connectivity. In *Proceedings of the 7th ACM international symposium on Mobile ad hoc networking and computing*, pages 131–142, 2006.

[3] A. J. Bernoff and C. M. Topaz. A primer of swarm equilibria. *SIAM Journal on Applied Dynamical Systems*, 10(1):212–250, 2011.

[4] A. L. Bertozzi, J. A. Carrillo, and T. Laurent. Blow-up in multidimensional aggregation equations with mildly singular interaction kernels. *Nonlinearity*, 22(3):683, 2009.

[5] A. L. Bertozzi, T. Kolokolnikov, H. Sun, D. Uminsky, and J. v. Brecht. Ring patterns and their bifurcations in a nonlocal model of biological swarms. *Communications in Mathematical Sciences*, 13(4):955–985, 2015.

[6] A. L. Bertozzi, T. Laurent, and J. Rosado. Lp theory for the multidimensional aggregation equation. *Communications on Pure and Applied Mathematics*, 64(1):45–83, 2011.

[7] M. R. D'Orsogna, Y.-L. Chuang, A. L. Bertozzi, and L. S. Chayes. Self-propelled particles with soft-core interactions: patterns, stability, and collapse. *Physical review letters*, 96(10):104302, 2006.

[8] L. Edelstein-Keshet, J. Watmough, and D. Grunbaum. Do travelling band solutions describe cohesive swarms? an investigation for migratory locusts. *Journal of mathematical biology*, 36(6):515–549, 1998.

[9] K. Fellner and G. Raoul. Stable stationary states of non-local interaction equations. *Mathematical Models and Methods in Applied Sciences*, 20(12):2267–2291, 2010.

[10] R. C. Fetecau, Y. Huang, and T. Kolokolnikov. Swarm dynamics and equilibria for a nonlocal aggregation model. *Nonlinearity*, 24(10):2681, 2011.

[11] A. Fijany and F. Vatan. A new method for sensor placement optimization. In *41st AIAA/ASME/SAE/ASEE Joint Propulsion Conference & Exhibit*, page 3715, 2005.

[12] V. Isaeva. Self-organization in biological systems. *Biology Bulletin*, 39(2):110–118, 2012.

[13] R. Kershner. The number of circles covering a set. *American Journal of mathematics*, 61(3):665–671, 1939.

[14] J.-E. Kim, M.-K. Yoon, J. Han, and C.-G. Lee. Sensor placement for 3-coverage with minimum separation requirements. In *International Conference on Distributed Computing in Sensor Systems*, pages 266–281. Springer, 2008.

[15] T. Laurent. Local and global existence for an aggregation equation. *Communications in Partial Differential Equations*, 32(12):1941–1964, 2007.

[16] R. Lukeman, Y.-X. Li, and L. Edelstein-Keshet. Inferring individual rules from collective behavior. *Proceedings of the National Academy of Sciences*, 107(28):12576–12580, 2010.

[17] I. Manring, J. H. Hill, G. Mohler, P. J. Brantingham, T. Williams, and B. White. Low-cost gunshot detection system with localization for community based violence interruption. In *2023 IEEE 10th International Conference on Data Science and Advanced Analytics (DSAA)*, pages 1–7, 2023.

[18] A. Mogilner, L. Edelstein-Keshet, L. Bent, and A. Spiros. Mutual interactions, potentials, and individual distance in a social aggregation. *Journal of Mathematical Biology*, 47(4):353–389, 2003.

[19] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, 2005.

[20] P. M. Morse. Diatomic molecules according to the wave mechanics. ii. vibrational levels. *Physical review*, 34(1):57, 1929.

[21] R. Olfati-Saber and R. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[22] W. Ren and E. Atkins. Distributed multi-vehicle coordinated control via local information exchange. *International Journal of Robust and Nonlinear Control*, 17(10-11):1002–1033, 2007.

[23] W. Ren and R. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

[24] D. Ruelle. *Statistical Mechanics: Rigorous Results*. G - Reference,Information and Interdisciplinary Subjects Series. World Scientific, 1999.

[25] C. M. Topaz, A. J. Bernoff, S. Logan, and W. Toolson. A model for rolling swarms of locusts. *The European Physical Journal Special Topics*, 157(1):93–109, 2008.

[26] C. M. Topaz and A. L. Bertozzi. Swarming patterns in a two-dimensional kinematic model for biological groups. *SIAM Journal on Applied Mathematics*, 65(1):152–174, 2004.

[27] J. von Brecht, T. Kolokolnikov, A. L. Bertozzi, and H. Sun. Swarming on random graphs. *Journal of Statistical Physics*, 151(1):150–173, 2013.

[28] J. H. von Brecht, B. Sudakov, and A. L. Bertozzi. Swarming on random graphs ii. *Journal of Statistical Physics*, 158(3):699–734, 2015.

[29] J. H. von Brecht and D. Uminsky. On soccer balls and linearized inverse statistical mechanics. *Journal of nonlinear science*, 22(6):935–959, 2012.

[30] J. H. Von Brecht, D. Uminsky, T. Kolokolnikov, and A. L. Bertozzi. Predicting pattern formation in particle interactions. *Mathematical Models and Methods in Applied Sciences*, 22(supp01):1140002, 2012.

[31] J. Woodworth, G. Mohler, A. L. Bertozzi, and P. Brantingham. Non-local crime density estimation incorporating housing information. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2028):20130403, 2014.

[32] F. Xiao and L. Wang. Dynamic behavior of discrete-time multiagent systems with general communication structures. *Physica A: Statistical Mechanics and its Applications*, 370(2):364–380, 2006.